Coulomb corrections and multiple e^+e^- pair production in ultra-relativistic nuclear collisions

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Abstract

We consider the problem of Coulomb corrections to the inclusive cross section. We show that these corrections in the limiting case of small charge number of one of the nuclei coincide with those to the exclusive cross section. Within our approach we also obtain the Coulomb corrections for the case of large charge numbers of both nuclei.

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In a set of recent publications the process of e^+e^- pair production in ultrarelativistic heavy-ion collisions was investigated by different groups of authors. The authors of [1, 2, 3] treated the nuclei as sources of the external field, and calculated the amplitude of the process at a fixed impact parameter using retarded solutions of the Dirac equation. After that the cross section was obtained by the integration over the impact parameter:

$$
\begin{aligned}
d\sigma &= \frac{m^2 d^3p\, d^3q}{(2\pi)^6 \varepsilon_p \varepsilon_q} \int d^2\rho \left| \int \frac{d^2k}{(2\pi)^2} \exp[i\mathbf{k}\boldsymbol{\rho}] \mathcal{M}\, \mathcal{F}_A(\mathbf{k}')\, \mathcal{F}_B(\mathbf{k}) \right|^2 \qquad (1)\\
\mathcal{M} &= \overline{u}(p) \left[\frac{\boldsymbol{\alpha}\ (\mathbf{k}-\mathbf{p}_\perp) + \gamma_0 m}{-p_+q_- - (\mathbf{k}-\mathbf{p}_\perp)^2 - m^2} \gamma_- + \frac{-\boldsymbol{\alpha}\ (\mathbf{k}-\mathbf{q}_\perp) + \gamma_0 m}{-p_-q_+ - (\mathbf{k}-\mathbf{q}_\perp)^2 - m^2} \gamma_+ \right] u(-q)\,.
\end{aligned}
$$

Here $\mathbf{p}$ and ε_p ($\mathbf{q}$ and ε_q) are the momentum and energy of the electron (positron), $u(p)$ and $u(-q)$ are positive- and negative-energy Dirac spinors, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\gamma_\pm = \gamma^0 \pm \gamma^z$, γ^μ are the Dirac matrices, $p_\pm = \varepsilon_p \pm p^z$, $q_\pm = \varepsilon_q \pm q^z$, m is the electron mass, $\mathbf{k}$ is a two-dimensional vector lying in the xy plane, $\mathbf{k}' = \mathbf{q}_\perp + \mathbf{p}_\perp - \mathbf{k}$, and the function $\mathcal{F}(\boldsymbol{\Delta})$ is proportional to the electron eikonal scattering amplitude in the Coulomb field:

$$
\mathcal{F}(\boldsymbol{\Delta}) = i\pi Z\alpha \frac{\Gamma(1-iZ\alpha)}{\Gamma(1+iZ\alpha)} \left(\frac{4}{\Delta^2}\right)^{1-iZ\alpha} \quad , \qquad (2)
$$

where $Z = Z_{A,B}$ is the charge number of the nucleus A, B. The nuclei A and B are assumed to move in the positive and negative directions of the z axis, respectively, and have the Lorentz factors $\gamma_{A,B} = 1/\sqrt{1-\beta_{A,B}^2}$. Using (1) the authors of [1, 2, 3] made the conclusion that the exact cross section coincides with that calculated in the Born approximation, i.e. in the lowest-order perturbation theory with respect to $\alpha Z_{A,B}$. The Born cross section can be obtained from (1) by the replacement $\mathcal{F}(\boldsymbol{\Delta}) \to \mathcal{F}^0(\boldsymbol{\Delta}) = 4i\pi Z\alpha/\Delta^2$. The statement about the absence of the Coulomb corrections was criticized in [4], where in the frame of the Weizsäcker-Williams approximation with respect to one of the nucleus the cross section of the process was expressed via the cross section of e^+e^- pair production by a photon in a Coulomb field [5]. As is well known, the latter contains the Coulomb corrections (higher order terms in $Z\alpha$). In our paper [6] we explicitly demonstrated that the following statements are true for the results obtained in [1, 2, 3]:

- the expression (1) actually contains the Coulomb corrections. The wrong conclusion on the absence of Coulomb corrections made in [1, 2, 3] is connected with illegal change of the order of integration in repeated integrals before the regularization of integrand.

- it cannot be applied for the calculation of the differential cross section with respect to both e^+ and e^-. The impossibility to interpret (1) as the cross section differential with respect to both particles is connected with the use of wave functions having the improper asymptotic behavior for the problem of pair production (retarded wave functions). This point was later realized by the authors of [7].

- the cross section (1) calculated in the lowest order in $Z_A\alpha$ (proportional to $(Z_A\alpha)^2$) and integrated over the momenta of at least one particle of the pair contains the Coulomb corrections in $Z_B\alpha$ which are in agreement with those obtained in the Weizsäcker-Williams approximation.

Though our paper [6] essentially clarified the situation with the Coulomb corrections, recently the paper [8] appeared. The authors of [8] still claim that the expression does not contain the Coulomb corrections. They also pointed out that the expression (1) makes sense only after the integration over the momenta $\mathbf{p}$ and $\mathbf{q}$ and gives the inclusive cross section σ_T of pair production being defined as

$$\sigma_T \equiv \int d^2\boldsymbol{\rho} \sum_{n=1}^{+\infty} nP_n \,, \tag{3}$$

where P_n is the probability to produce exactly n pairs in a collision at impact parameter $\boldsymbol{\rho}$. Note that the usual definition of the inclusive cross section, as a sum of cross sections of all possible processes, is different:

$$\sigma_{incl} \equiv \int d^2\boldsymbol{\rho} \sum_{n=1}^{+\infty} P_n \,. \tag{4}$$

The probabilities P_n are calculated exactly in the external field and their sum in the r.h.s. of (4) is expressed via the vacuum-to-vacuum transition probability P_0 as $\sum_{n=1}^{\infty} P_n = 1 - P_0$.

The cross section σ_T differs from the exclusive cross section σ_1 of the production of exactly one pair

$$\sigma_1 \equiv \int d^2\boldsymbol{\rho}\, P_1 \tag{5}$$

The authors of [8] suppose that this circumstance justifies the absence of the Coulomb corrections in their result.

In the present paper we consider the problem of Coulomb corrections to σ_T and demonstrate their existence.

Let us show first that difference in the definition of σ_T and σ_1 can not justify the absence of the Coulomb corrections in the former. Indeed, the expansion of the probability P_n in the parameters $Z_A\alpha$ and $Z_B\alpha$ starts from the term, proportional to $(Z_A\alpha)^{2n}(Z_B\alpha)^{2n}$. Therefore the terms $\propto (Z_A\alpha)^2(Z_B\alpha)^{2l}$ with $l \geq 1$ are contained only in P_1. Therefore, the terms, quadratic in $Z_A\alpha$, in σ_T and σ_1 should coincide. By means of the Weizsäcker-Williams approximation with respect to the nucleus A it is easy to understand that the term in σ_1, quadratic in $Z_A\alpha$, contains the Coulomb corrections in the parameter $Z_B\alpha$.

Now we pass to the explicit calculations of σ_T. For the sake of simplicity we consider the process in the frame where both nuclei have the same Lorentz factor $\gamma_A = \gamma_B = \gamma = 1/\sqrt{1-\beta^2}$. First of all it is worth noting that the integration of (1) with respect to $\mathbf{p}, \mathbf{q}$ and $\boldsymbol{\rho}$ leads to the logarithmic divergence. Of course, this is the consequence of setting $\beta = 1$ in the light-front approach used in [1, 2, 3]. In more accurate approach the regularizing terms proportional to $1/\gamma^2$ should be kept in denominators. In particular, this leads to the regularization of the integrals over p_z and q_z, which is equivalent (with the logarithmic accuracy) to the integration over these variables from $-m\gamma$ to $m\gamma$. For this momenta the velocities of both particles in pair are less than those of the nuclei.

Since the expression (1) is ill-defined, it is possible to perform the mathematical transformations of it only after the regularization. Assuming this regularization to be made in $\mathcal{M}$ and $\mathcal{F}$ we take the integral over $\boldsymbol{\rho}$. As a result we have for σ_T:

$$\sigma_T \;=\; \iint \frac{m^2 d^3p\, d^3q}{(2\pi)^6\varepsilon_p\varepsilon_q}\frac{d^2k}{(2\pi)^2}|\mathcal{M}|^2|\mathcal{F}_A(\mathbf{k}')|^2|\mathcal{F}_B(\mathbf{k})|^2 \tag{6}$$

If one substitutes non-regularized $\mathcal{F}$ from (2) into (6) then the Coulomb corrections cancel. However, this substitution is illegal since it leads to the divergence. It was the source of mistake made in [1, 2, 3, 8].

To proceed with the calculations it is convenient to split the expression (6) as

$$\sigma_T = \sigma^b + \sigma^c_A + \sigma^c_B + \sigma^c_{AB}\,, \tag{7}$$

where

$$\left.\begin{array}{l}\sigma^b\\ \sigma^c_B\\ \sigma^c_{AB}\end{array}\right\} = \iint \frac{m^2 d^3p\, d^3q}{(2\pi)^6\varepsilon_p\varepsilon_q}\frac{d^2k}{(2\pi)^2}|\mathcal{M}|^2 \times \left\{\begin{array}{l}|\mathcal{F}^0_A(\mathbf{k}')|^2|\mathcal{F}^0_B(\mathbf{k})|^2\\ |\mathcal{F}^0_A(\mathbf{k}')|^2\left[|\mathcal{F}_B(\mathbf{k})|^2-|\mathcal{F}^0_B(\mathbf{k})|^2\right]\\ \left[|\mathcal{F}_A(\mathbf{k}')|^2-|\mathcal{F}^0_A(\mathbf{k}')|^2\right]\left[|\mathcal{F}_B(\mathbf{k})|^2-|\mathcal{F}^0_B(\mathbf{k})|^2\right]\end{array}\right. \tag{8}$$

The term σ_A^c is obtained from σ_B^c by obvious substitution. Here $\mathcal{F}^0(\mathbf{\Delta})$, as well as $\mathcal{F}(\mathbf{\Delta})$, is assumed to be regularized in a proper way. In (7) the term σ^b is the Born part of σ_T, σ_A^c and σ_B^c contain the terms proportional to $(Z_B\alpha)^2(Z_A\alpha)^{2n}$ and $(Z_A\alpha)^2(Z_B\alpha)^{2n}$, respectively, $n \geq 2$. At last, σ_{AB}^c contains the terms proportional to $(Z_A\alpha)^n(Z_B\alpha)^l$ with $n, l > 2$.

Let us discuss now the regularization. Note that the expression (1) was derived without using specific character of the Coulomb potential. For arbitrary potential $V(r)$ we have

$$\begin{aligned} \mathcal{F}(\mathbf{\Delta}) &= \int d^2\rho \exp[-i\boldsymbol{\rho}\mathbf{\Delta}]\left\{\exp[-i\chi(\rho)] - 1\right\} , \\ \chi(\rho) &= \int\limits_{-\infty}^{\infty} dz V\left(\sqrt{z^2+\rho^2}\right) . \end{aligned} \tag{9}$$

The eikonal phase $\chi(\rho)$ is finite if $rV(r) \to 0$ at $r \to \infty$. Moreover, under this restriction on the potential we obtain the finite result for σ_T. As known, the correct expression for $\mathcal{F}^0_{A,B}(\mathbf{\Delta})$ for $\beta < 1$ is

$$\mathcal{F}^0_{A,B}(\mathbf{\Delta}) = \frac{4i\pi Z_{A,B}\alpha}{\Delta^2 + a_\pm^2}, \quad a_\pm = (p_\pm + q_\pm)/2\gamma . \tag{10}$$

These expressions for $\mathcal{F}^0_{A,B}(\mathbf{\Delta})$ correspond to the choice of effective potential $V_{A,B}(r) = -Z_{A,B}\alpha \exp\left[-ra_\pm\right]/r$. The quantity $\mathcal{F}$ for this potential reads

$$\mathcal{F}_{A,B}(\mathbf{\Delta}) = 2\pi \int d\rho\rho J_0(\rho\Delta)\left\{\exp[2iZ_{A,B}\alpha K_0(\rho a_\pm)] - 1\right\} , \tag{11}$$

where J_0 is the Bessel function and K_0 is the modified Bessel function of the third kind. Let us emphasize that the regularization of $\mathcal{F}$ in (11) is not reduced to the substitution $\mathbf{\Delta}^2 \to \mathbf{\Delta}^2 + a_\pm^2$ in (2) which was suggested in [2].

For all terms in σ_T the main contribution to the integrals comes from the region of integration

$$|\mathbf{k}|,\, |\mathbf{k}'| \ll m , \qquad |p_z|,\, |q_z| \ll m\gamma , \qquad |\mathbf{p}_\perp - \mathbf{q}_\perp| \sim m .$$

According to the first restriction we can expand $\mathcal{M}$ with respect to both $\mathbf{k}$ and $\mathbf{k}'$. Due to the gauge invariance the first non-zero term of this expansion reads $\mathcal{M} = k_i k'_j M_{ij}$. Passing to the variables $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{r} = (\mathbf{p}_\perp - \mathbf{q}_\perp)/2$, we obtain from (8):

$$\left.\begin{matrix} \sigma^b \\ \sigma_A^c \\ \sigma_B^c \\ \sigma_{AB}^c \end{matrix}\right\} = \iint \frac{m^2 dp_z\, dq_z d^2r}{4(2\pi)^4 \varepsilon_p \varepsilon_q} |M_{ij}|^2 \left\{\begin{matrix} L_A L_B \\ G_A L_B \\ L_A G_B \\ G_A G_B \end{matrix}\right. \tag{12}$$

Here

$$L_{A,B} = \int\limits_{|\mathbf{k}|<m} \frac{d^2k}{(2\pi)^2} k^2 |\mathcal{F}^0_{A,B}(\mathbf{k})|^2 = 8\pi (Z_{A,B}\alpha)^2 \ln(m/a_\pm)\,, \tag{13}$$

$$G_{A,B} = \int \frac{d^2k}{(2\pi)^2} k^2 \left[|\mathcal{F}_{A,B}(\mathbf{k})|^2 - |\mathcal{F}^0_{A,B}(\mathbf{k})|^2 \right] .$$

The functions $G_{A,B}$ have been calculated in our paper [6]. It was shown in [6] that, though the main contribution to the integral over $\mathbf{k}$ in $G_{A,B}$ comes from the region $k \sim a_\pm \ll m$, where $|\mathcal{F}^0_{A,B}(\mathbf{k})|$ differs from $|\mathcal{F}_{A,B}(\mathbf{k})|$, the quantities G_A and G_B are universal functions of $Z_A\alpha$ and $Z_B\alpha$, respectively. They have the form

$$G_{A,B} = -8\pi (Z_{A,B}\alpha)^2 [\mathrm{Re}\,\psi(1 + iZ_{A,B}\alpha) + C] = -8\pi (Z_{A,B}\alpha)^2 f(Z_{A,B}\alpha)\;, \tag{14}$$

where C is the Euler constant, $\psi(x) = d\ln\Gamma(x)/dx$.

Straightforward calculation leads to the following expression for $|M_{ij}|^2$:

$$|M_{ij}|^2 = \frac{8}{m^2(r^2+m^2)(p_-+q_-)(p_++q_+)} \left[1 - \frac{2(r^4+m^4)}{(r^2+m^2)(p_-+q_-)(p_++q_+)} \right]\;, \tag{15}$$

where $p_\pm = \sqrt{p_z^2 + r^2 + m^2} \pm p_z$ and similar to $q_\pm$. At the derivation of (15) we have performed the summation over the polarizations of both particles in pair. Substituting (15) in (12) and performing the integration with the logarithmic accuracy, we obtain for σ^b and $\sigma^c_{A,B}$:

$$\sigma^b = \frac{28 (Z_A\alpha)^2 (Z_B\alpha)^2}{27\pi m^2} \ln^3(\gamma^2)\,, \tag{16}$$

$$\sigma^c_A = -\frac{28 (Z_A\alpha)^2 (Z_B\alpha)^2}{9\pi m^2} f(Z_A\alpha) \ln^2(\gamma^2)\,,$$

$$\sigma^c_B = -\frac{28 (Z_A\alpha)^2 (Z_B\alpha)^2}{9\pi m^2} f(Z_B\alpha) \ln^2(\gamma^2)\,.$$

As expected, these results agree with those obtained with the use of the Weizsäcker-Williams approximation. Within our approach we also get the following expression for σ^c_{AB}:

$$\sigma^c_{AB} = \frac{56 (Z_A\alpha)^2 (Z_B\alpha)^2}{9\pi m^2} f(Z_A\alpha) f(Z_B\alpha) \ln(\gamma^2) \tag{17}$$

In the arbitrary frame one should replace $\gamma^2 \to \gamma_A\gamma_B$ in (16) and (17).

Thus we demonstrated that the difference in definitions of the exclusive cross section σ_1 and the inclusive cross section σ_T can not lead to the cancellation of the Coulomb corrections in the latter. Using the proper regularization of the expression for σ_T we obtained the result

for Coulomb corrections which in the limiting case $Z_A\alpha \ll 1$ (or $Z_B\alpha \ll 1$) agrees with that obtained in the Weizsäcker-Williams approximation.

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